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Asymptotic analysis of solutions to parabolic systems

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Abstract We study asymptotics as $t \to \infty$ of solutions to a linear, parabolic system of equations with time-dependent coefficients in $\Omega \times (0, \infty)$, where Ω is a bounded domain. On $\partial\Omega \times (0, \infty)$ we prescribe the homogeneous Dirichlet boundary condition. For large values of t, the coefficients in the elliptic part are close to time-independent coefficients in an integral sense which is described by a certain function $\kappa(t)$. This includes in particular situations when the coefficients may take different values on different parts of Ω and the boundaries between them can move with t but stabilize as $t \to \infty$. The main result is an asymptotic representation of solutions for large t. A consequence is that for $\kappa \in L^1(0, \infty)$, the solution behaves asymptotically as the solution to a parabolic system with time-independent coefficients.

Keywords Asymptotic behaviour · Parabolic system · Spectral splitting · Perturbation

1 Introduction

Let Ω denote an open, bounded region in $\mathbb{R}^n$ with Lipschitz boundary and introduce $Q = \Omega \times (0, \infty)$. By $x = (x_1, \ldots, x_n)$ we denote the variables in Ω and by t the unbounded variable. We consider the parabolic system

$$u_t - \sum_{i,j=1}^{n} (A_{ij} u_{x_j})_{x_i} + Au = 0 \quad \text{in } Q, \tag{1}$$

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where $u = (u_1, \ldots, u_N)$ is a function from Q to $\mathbb{C}^N$ and $A_{ij}, i, j = 1, \ldots, n$, and A are quadratic matrices of size $N \times N$ whose elements are measurable functions on Q. We assume that u satisfies the Dirichlet boundary condition

$$u(x, t) = 0 \quad \text{if } x \in \partial\Omega,\ t > 0 \tag{2}$$

and that

$$u(x, 0) = \psi(x), \tag{3}$$

where ψ is a function from $\left(L^2(\Omega)\right)^N$.

For general theory of parabolic equations and systems, which include in particular solvability and uniqueness results, we refer to Ladyženskaja et al. [6], Dautray, Lions [1], Lions, Magenes [7], Friedman [4] and Eidel'man [2]. Evolution problems of the above type appear for example in biology and chemistry when studying reaction diffusion problems, see for example Murray [8] or Fife [3]. We are concerned only with the asymptotic behaviour of solutions as $t \to \infty$. Therefore, we suppose that the matrices A_{ij} and A can be written as

$$A_{ij}(x, t) = A_{ij}^{(0)}(x) + A_{ij}^{(1)}(x, t)$$

and

$$A(x, t) = A^{(0)}(x) + A^{(1)}(x, t),$$

where $A_{ij}^{(1)}$ and $A^{(1)}$ are considered as perturbations. We require that the quantity

$$\kappa(t) = \sum_{i,j=1}^{n} \|A_{ij}^{(1)}\|_{L^{s_1,2}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^{s_2,1}(\mathcal{C}_t)} \tag{4}$$

is small enough, where

$$\mathcal{C}_t = \Omega \times (t, t+1)$$

and s_1 and s_2 are introduced later. We also assume some boundedness and symmetry conditions on the matrices A_{ij}, $A_{ij}^{(0)}$, A and $A^{(0)}$. Under these assumptions, we have derived an asymptotic representation for u which is presented in Theorem 1.

The asymptotic behaviour of solutions to (1) has been studied under the assumption that the L^∞-norms of $A_{ij}^{(1)}$ and $A^{(1)}$ are small for large t. This does not cover all physically relevant situations, for example when the coefficients take different values on different parts of Ω and the boundaries between them can move with t but stabilize as $t \to \infty$. As far as we know, the situation when the coefficients are small only in the integral sense described by κ has not been investigated before.

2 Problem formulation and results

We assume that the relation $\left(A_{ij}^{(0)}\right)^* = A_{ji}^{(0)}$, where A^* denotes the adjoint matrix of A, holds for every pair (i, j) and that $\left(A^{(0)}\right)^* = A^{(0)}$. The matrices $A_{ij}^{(0)}$ fulfill the two-sided inequality

$$\nu \sum_{i=1}^{n} |\xi_i|^2 \le \sum_{i,j=1}^{n} (A_{ij}^{(0)}\xi_j, \xi_i) \le \nu^{-1} \sum_{i=1}^{n} |\xi_i|^2 \tag{5}$$

for all $\xi_i, \xi_j \in \mathbb{C}^N$ and some positive constant ν. Here we use the notations $(u, v) = u_1\overline{v_1} + \ldots + u_N\overline{v_N}$ and $|u| = (u, u)^{1/2}$ for $u, v \in \mathbb{C}^N$. The matrix $A^{(0)}$ is supposed to belong to $\left(L^q(\Omega)\right)^{N^2}$, where

$$\begin{cases} q \in [n, \infty] & \text{if } n \geq 3 \\ q \in (2, \infty] & \text{if } n = 2 \\ q \in [2, \infty] & \text{if } n = 1. \end{cases} \tag{6}$$

Writing $A^{(0)} = A_+^{(0)} - A_-^{(0)}$, where both matrices $A_+^{(0)}$ and $A_-^{(0)}$ are positive, we require that $A_-^{(0)}$ is bounded. This means that there exists a constant ν_1 such that

$$\| |A_-^{(0)}| \|_{L^\infty(\Omega)} \leq \nu_1, \tag{7}$$

where $|A_-^{(0)}(x)|$ denotes the matrix norm of $A_-^{(0)}(x)$.

Under the above conditions on the matrices $A_{ij}^{(0)}$ and $A^{(0)}$, the operator $L^{(0)} : \left(W_0^{1,2}(\Omega)\right)^N \to \left(W^{-1,2}(\Omega)\right)$, defined as

$$L^{(0)}u = -\sum_{i,j=1}^{n} (A_{ij}^{(0)} u_{x_j})_{x_i} + A^{(0)}u,$$

has a discrete spectrum with the only limit point ∞. Let us denote by

$$\lambda_1 = \ldots = \lambda_J < \lambda_{J+1} \leq \lambda_{J+2} \leq \ldots$$

its eigenvalues and by $\phi_1, \phi_2, \ldots$ its corresponding eigenfunctions. They can be chosen in such way that they form an ON-basis for $\left(L^2(\Omega)\right)^N$. Observe that J denotes the multiplicity of the first eigenvalue. The conditions on $A_{ij}^{(0)}$, $A^{(0)}$ and Ω imply that there exists a $p \in (2, \infty]$ such that $\phi_k \in \left(W^{1,p}(\Omega)\right)^N$, $k = 1, 2, \ldots$.

We also assume some similar conditions on A_{ij} and A. Namely, the relations $A_{ij}^* = A_{ji}$, $i, j = 1, \ldots, n$, hold and the matrix A is hermitian. We assume further that (5) also is valid with $A_{ij}^{(0)}$ replaced by A_{ij} and that $A \in \left(L_{\text{loc}}^{q,r}(Q)\right)^{N^2}$, where q is the same as in (6) and $r = 2q/(2q - n)$. Furthermore, writing $A = A_+ - A_-$, where A_+ and A_- are positive, we assume that

$$\| |A_-| \|_{L^\infty(Q)} \leq \nu_1,$$

where ν_1 is the same as in (7).

The main characteristic of our perturbation is the function κ given by (4), where $s_1 = 2p/(p-2)$ and $s_2 = 2$ if $n < p$, s_2 is an arbitrary number in $(2, n]$ if $n = p$ and $s_2 = 2np/(np - 2n + 2p)$ if $n > p$. Since $A^{(1)} \in \left(L_{\text{loc}}^{q,r}(Q)\right)^{N^2}$ and $A_{ij}^{(1)}$ is bounded, it follows that $\kappa(t)$ is finite for every $t \geq 0$. We set

$$\kappa_0 = \sup_{t \geq 0} \kappa(t),$$

and consider perturbations subject to

$$\kappa_0 \leq \varkappa, \tag{8}$$

where $\varkappa$ is a sufficiently small constant depending on $n, N, \Omega, A_{ij}^{(0)}, A^{(0)}, p, s_2, \nu$ and ν_1. An exact value of $\varkappa$ is difficult to give but the requirement is that $\varkappa$ is so small that some inequality type conditions appearing in the proof of Theorem 1 are satisfied. Note that $\varkappa$ does not depend on $A_{ij}^{(1)}$ or $A^{(1)}$.

We let ∇ denote the gradient with respect to the x-variables and define $\left(V_{0,\mathrm{loc}}^2(Q)\right)^N$ as the space consisting of vector functions u such that

$$|u|_{\mathcal{C}_t} = \operatorname*{ess\,sup}_{t<s<t+1} \|u(\cdot, s)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\mathcal{C}_t)}$$

is finite for every $t \geq 0$. It can be proved that problem (1)–(3) has a unique solution in $\left(V_{0,\mathrm{loc}}^2(Q)\right)^N$. The main result of our work is the following theorem.

Theorem 1 *If the constant $\varkappa$ introduced in (8) is small enough, then the unique solution u in $\left(V_{0,\mathrm{loc}}^2(Q)\right)^N$ of (1)–(3) can be represented as*

$$u(x,t) = e^{-\lambda_1 t + \int_0^t (-f(s)+\Lambda(s))\,ds}\left(w_0 \sum_{k=1}^{J} \theta_k(t)\phi_k(x) + V(x,t)\right), \tag{9}$$

where w_0 is a constant, $\Theta = (\theta_1, \ldots, \theta_J)$ is an absolutely continuous unit vector, Λ is a function belonging to $L^1_{\mathrm{loc}}(0,\infty)$ and

$$f = (\mathcal{R}\Theta, \Theta). \tag{10}$$

Here, $\mathcal{R}$ denotes the $J \times J$ matrix with entry (k,l) equal to

$$\mathcal{R}_{kl} = \int_\Omega \left[\sum_{i,j=1}^{n} (A_{ij}^{(1)}\phi_{l\,x_j}, \phi_{k\,x_i}) + (A^{(1)}\phi_l, \phi_k)\right] dx.$$

Furthermore, the following estimates are valid:

$$|w_0| \leq C\|\psi\|_{L^2(\Omega)}$$

and

$$\|\Lambda\|_{L^1(t,t+1)} \leq C\kappa(t)\left(\int_{-1}^{t} e^{-b_0(t-s)}\kappa(s)\,ds + \kappa(t)\right),$$
$$\|\Theta'\|_{L^1(t,t+1)} \leq C\kappa(t),$$
$$|V|_{\mathcal{C}_t} \leq C\|\psi\|_{L^2(\Omega)}\left(e^{-b_0 t} + \int_{-1}^{t} e^{-b_0(t-s)}\kappa(s)\,ds + \kappa(t)\right)$$

for $t \geq 0$. Here, $b_0 = \lambda_{J+1} - \lambda_1 - C_1\kappa_0$ and C and C_1 denote constants depending on $n, N, \Omega, A_{ij}^{(0)}, A^{(0)}, p, s_2, \nu$ and ν_1. In (4) we extend $A_{ij}^{(1)}(x,\tau)$ and $A^{(1)}(x,\tau)$ by 0 for $\tau < 0$.

As a consequence, the asymptotic formula (9) implies the estimate

$$|u|_{\mathcal{C}_t} \leq C_1\|\psi\|_{L^2(\Omega)} e^{-\lambda_1 t + \int_0^t (-f(s)+C_2\kappa(s)^2)\,ds}.$$

If, in addition, $\kappa \in L^1(0,\infty)$, then

$$u(x,t) = e^{-\lambda_1 t}\left(\sum_{k=1}^{J} b_k\phi_k(x) + \omega(x,t)\right),$$

where $b_k, k = 1, \ldots, J$, are constants which may depend on $A_{ij}^{(1)}, i, j = 1, \ldots, n, A^{(1)}$ and ψ and $|\omega|_{\mathcal{C}_t} \to 0$ as $t \to \infty$. We have here the same leading term as in the case when $A_{ij}^{(1)} = 0, i, j = 1, \ldots, n, A^{(1)} = 0$. If, instead, $A^{(1)} = 0$,

$$\sum_{i,j=1}^{n} \int_0^{\infty} \int_{\Omega} |A_{ij}^{(1)}(x,t)|\, dx\, dt < \infty$$

and $p = \infty$, i.e. the gradients of the eigenfunctions belong to $L^{\infty}(\Omega)$, then we obtain

$$u(x,t) = e^{-\lambda_1 t}\left(b \sum_{k=1}^{J} \theta_k(t)\phi_k(x) + \omega(x,t)\right),$$

where $|\omega|_{\mathcal{C}_t} \to 0$ as $t \to \infty$ and b is a constant which may depend on $A_{ij}^{(1)}, i, j = 1, \ldots, n$, and ψ.

As can be seen from (10), the function f in (9) is not given exactly since its definition contains the unknown vector-valued function Θ. If the eigenvalue λ_1 is simple, i.e. $J = 1$, and A and A_{ij} are real-valued, then $\Theta = 1$ and we arrive at the following asymptotic expansion for u:

$$u(x,t) = e^{-\lambda_1 t + \int_0^t (-\mathcal{R}(s) + \Lambda(s))\, ds}\big(w_0\phi_1(x) + V(x,t)\big).$$

The proofs of Theorem 1 and the subsequent statements can be found in Rand [9], Paper 2.

Ordinary differential equations with unbounded operator coefficients which include parabolic ones are studied in Kozlov, Maz'ya [5]. In particular, asymptotic results from Part III in [5] can give the asymptotic formula (9) under the restriction that λ_1 is simple and the quantity

$$\sum_{i,j=1}^{n} \|A_{ij}^{(1)}\|_{L^{\infty}(\mathcal{C}_t)} + \|\rho^{-2}A^{(1)}\|_{L^{\infty}(\mathcal{C}_t)},$$

where $\rho(x)$ denotes the distance to $\partial\Omega$, is small.

The proof of Theorem 1 can very briefly be outlined in the following way. Using spectral splitting, we write

$$u(x,t) = \sum_{k=1}^{J} h_k(t)\phi_k(x) + w(x,t), \tag{11}$$

where $h_k = \int_{\Omega}(u, \phi_k)\, dx$ and $w(x,t)$ is the remainder term. The most part of the proof is devoted to derivation of a system of first order ordinary differential equations for $h_1, \ldots, h_J$ perturbed by a small integro-differential term and to estimation of w. An important role here plays a preliminary spectral splitting with J in (11) replaced by M, where M is sufficiently large. After this, the proof is completed by study of asymptotic behaviour of solutions to the perturbed system of ordinary differential equations.

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